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17.3 Production planning for non-cooperating companies with nonlinear optimization

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Abstract:
We consider a production planning problem where two competing companies are selling their items on a common market. Moreover, the raw material used in the production is a limited non-renewable resource. The revenue per item sold depends on the total amount of items produced by both players. If they collaborate they could apply a production strategy that leads to the highest combined revenue. Usually the formation of such syndicates is prohibited by law; hence we assume that one company does not know how much the other company will produce. We formulate the problem for company A to find an optimal production plan without information on the strategy of company B as a nonlinear mathematical optimization problem. In its naive formulation the model is too large, making its solution practically impossible. After a reformulation we find a much smaller model, which we solve by spatial branch-and-cut methods and linear programming. We discuss the practical implications of our solutions.

Keywords:
Non-Cooperative Two-Person Games; Mixed-Integer Nonlinear Optimization

1 INTRODUCTION

Economists, politicians and entrepreneurs alike have been concerned with nonrenewable resources for a long time. Nonrenewable resources (e.g. oil, iron, zinc, phosphate, etc.) share the characteristic that they cannot be replenished within a relevant time frame. The economic literature on nonrenewable resources started with the work of Harold Hotelling [1]. Hotelling examined the case of a fully competitive market where extracting a marginal unit of the resource was costless. He showed that in this particular case, the price of the nonrenewable resource should increase at the rate of interest. More generally, it is the *shadow price* of the resource that should increase at the rate of interest. Both the specific and the more general result have become known in the literature as the *Hotelling rule*. Since Hotelling's seminal work, a large number of papers in economics have studied the Hotelling rule both mathematically and empirically under various assumptions (see e.g. Krautkraemer [2], Gaudet [3], or Kronenberg [4] for reviews of the literature).

A critical assumption in each of these studies is that producers of the nonrenewable resource act rationally in the economic sense. This entails that producers are -inter alia- assumed to maximize long run profits and are assumed to be able to do so in the optimal way. In the typical case when there are multiple active producers on the market, economic rationality means the outcome of the market can be characterized by a Nash equilibrium. In the context of nonrenewable resources, the Nash equilibrium entails that each producer has adopted the best possible (profit-maximizing) production strategy conditional on the *equilibrium* production strategy of all the other producers on the market.

However, in practice, many producers may not be following the Nash equilibrium production strategy. For example, the president of an oil producing nation may be more concerned about being re-elected than about maximizing long-term oil profits. In this case, he may not follow the Nash equilibrium production strategy but may instead produce at capacity in every period, to maximize short-term profits. A similar argument can be made for the CEO of a resource firm, whose bonus structure is unlikely to be based on long term profits. When some producers do not optimize, it will no longer be optimal to choose the Nash equilibrium production strategy even for producers who do optimize. Instead, they should maximize their profits given the range of possible production strategies they think the other producers on the market will adopt.

This article derives the profit maximizing strategy for producers who are faced with non-optimizing competitors. In particular, we will look at a market with two producers where one producer randomly chooses a production strategy from the set of possible production strategies. This approach is similar to the cognitive hierarchy (or level K) approach used in behavioral economics (see e.g., Camerer, Ho and Chong [5]). (Cognitive hierarchy theory assumes that different players

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have a different level of rationality, where level 1 best responds to level 0, level 2 best responds to level 1 (or a mix of level 0 and level 1), etc. The behavior of level zero players is typically assumed to be uniformly random; we will use a similar approach in this study.) We then solve for the optimal production strategy for the other producer on the market using spatial branch-out methods and linear programming.

The remainder of this article is structured as follows. In Section 2 we introduce the model that defines the players' payoff function, that rational players aim to maximize. We start in Section 3 with an analysis of the cooperative case, i.e., both players are able to communicate and thus are able to maximize their income. In general, forming such a monopoly would be not allowed, and hence we continue in Section 4 with the case that one player has to find a strategy to beat his competitor, irrespective of how the other player behaves. We discuss our results in Section 5. Conclusions are drawn in Section 6.

2 THE MODEL

We consider a two player game, where both players A and B make one simultaneous move for $n_t = 6$ consecutive rounds. Each player represents a producer, selling from its limited and nonrenewable product stock on a duopolistic market. In each round, each player has to decide on the number of products to sell. The price p that the players earn for one unit of their product depends on the total number of products offered by both players, and is computed according to the linear equation

$$p_t = \left(a - b(q_A^t + q_B^t)\right), \quad t \in \{1, 2, \ldots, n_t\}, \tag{1}$$

where a and b are parameters, q_A^t, q_B^t are the quantities sold by player A and player B respectively, and t is the current round. As abbreviation, we write q_i for the vector $(q_i^1, \ldots, q_i^{n_t})$ for $i \in \{A, B\}$. In each round, each player receives interest on the cumulative income of the previous rounds with an interest rate of $r > 0$. The values we use for our numerical studies are shown in Table 1.

After n_t rounds, the cumulative income from selling the product and from accumulating interest is added up, to yield the final payout $x_i(q_A, q_B)$ for player $i \in \{A, B\}$:

$$\begin{aligned} x_i(q_A, q_b) &= \sum_{t=1}^{n_t} \left(q_i^t \cdot p_t \cdot (1+r)^{n_t - t}\right) \\ &= \sum_{t=1}^{n_t} \left(q_i^t \cdot \left(a - b(q_A^t + q_B^t)\right) \cdot (1+r)^{n_t - t}\right). \end{aligned} \tag{2}$$

For a fixed value of q_B^t, the revenue of player A in round t, defined by $q_A^t \mapsto q_A^t \cdot p_t$, has a maximum at $\frac{1}{2b}(a - bq_B^t)$.

In Figure 1, the revenues $(q_A^t \cdot p_t)$ of player A for given values of q_A^t and q_B^t are represented in a contour plot (for t fixed). For a given value of q_B^t, there is one unique q_A^t with the highest possible revenue (shown by the red dashed line). Selling a larger number of products than this optimum will actually result in a *lower* revenue while selling *more* of the product in stock.

This property of the game leads to a high potential of conflicting strategies between the two players. It follows, that a good strategy must to be robust against unexpected decisions of the opponent.

The task of each player $i \in \{A, B\}$ is to choose quantities, such that his final payout x_i is maximized. The cases of continuous as well as integer quantities q_i^t can be considered. In practice, this will depend on whether the quantity is non-divisible (for example cars or cell phones) or divisible (for example raw materials).

The following criteria for a successful strategy can be derived from these equations:

- i Distributing the quantities evenly is preferable to selling everything at once, due to the decreasing price depending on the amount of product on the market.
- ii Having a high income in the earlier rounds is preferable to having a high income in the later rounds, due to the higher achieved interest.
- iii Since there is no information on the behavior of the opponent in the current round, the strategy should be as robust as possible against the decisions of the opponents.

Strategies (i) and (ii) are obviously contradictory, and also strategy (iii) can be in contradiction to strategies (i) and (ii), even though this is harder to quantify. Finding the best compromise between these criteria constitutes the desired solution that we aim to compute.

Letter	Parameter Name	Value
n_t	Number of rounds	6
r	Interest rate	.1
a	Maximum Price	372
b	Slope of price decay	1
s	Per player resource stock	170

Table 1: Overview of model parameters

3 COOPERATIVE CASE

To get a better understanding of the model, we first consider the cooperative case, where both players communicate in order to find the strategy that maximizes their combined income, and then strictly stick to this strategy over the time horizon. To find the highest possible combined income, we formulate the following nonlinear optimization problem:

$$\begin{aligned} &\underset{q_A, q_B}{\text{maximize}} \quad x_A(q_A, q_B) + x_B(q_a, q_B), \\ &\text{subject to} \quad \sum_{t=1}^{n_t} q_i^t \leq s, \quad \forall i \in \{A, B\}. \end{aligned} \tag{3}$$

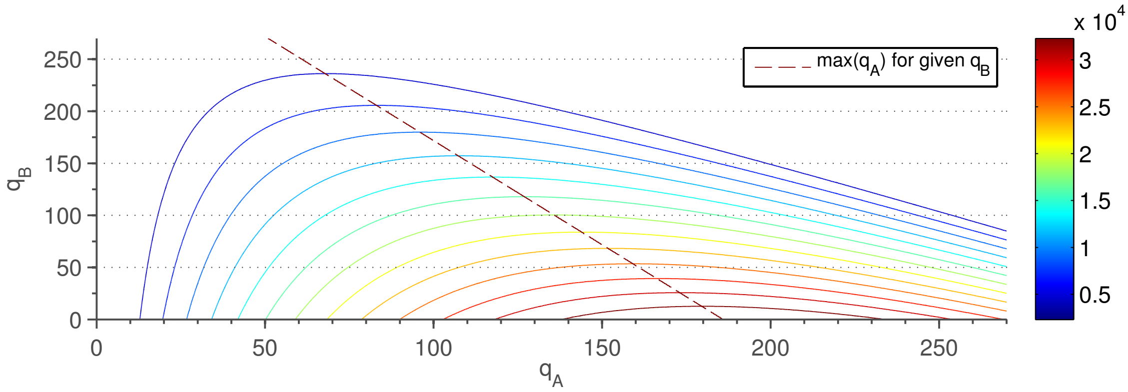

Figure 1: Contour plot of possible revenues of player A in a single round t for values of q_A^t and q_B^t. The broken, dark red line connects the highest possible revenues for given values of q_B^t, defined by $q_B^t = 372 - 2q_A^t$.

Since the final payout depends only on the sum of the quantities sold by the two players in each round, we introduce new variables q_c^t for the combined production and substitute $q_A^t + q_B^t = q_c^t$ (for $t = 1, 2, \ldots, n_t$) in the model formulation (3). This leads to a reduction of the size of the solution space, and thus a significant reduction of the solution time.

In a first experiment we set the per player resource limit to infinity, $s = \infty$. Then we receive the solution shown in Table 2 for q_c with a combined final payout of $2.669 \cdot 10^5$.

q_c^1	q_c^2	q_c^3	q_c^4	q_c^5	q_c^6
186	186	186	186	186	186

Table 2: Optimal solution for the cooperative case with $s = \infty$.

Since our reduced setup eliminates the motivations behind strategies (ii) and (iii), the optimal strategy is governed entirely by (i). We can argue that this is indeed the optimal solution, since it is the maximum of the income function in $q_A^t + q_B^t$ at each point in time $t = 1, \ldots, n_t$. Incidentally, this maximum is found at an integer quantity value, so the solution is optimal in the continuous as well as the integer case. In order to obtain a solution in the variables q_A^t and q_B^t from the values of q_c^t, we need to split the total production quantity among the two players A and B. In the integer case, there are $186^6 \approx 4.1 \cdot 10^{13}$ possible choices for q_A, q_B together that all sum up to the solution q_c and consequently have the same objective value. In the continuous case there are infinitely many distributions.

As a numerical solver for the nonlinear optimization problems we use SCIP. Information on the MINLP framework SCIP can be found in Achterberg [6], and in particular on nonlinear aspects of SCIP in Berthold, Heinz, and Vigerske [7]. Setting the per player resource stock to $s = 170$, we receive the continuous and integer solutions shown in Table 3.

	q_c^1	q_c^2	q_c^3	q_c^4	q_c^5	q_c^6
C	85.43	75.37	64.30	52.13	38.75	24.02
I	86	75	64	52	39	24

Table 3: Optimal solutions for the cooperative continuous (C) and integer (I) case.

We emphasize that the integer solution cannot be obtained by simply rounding the continuous solution to the nearest integer values ($q_c^1 = 85.43$ in the continuous case must be rounded up to $q_c^1 = 86$ in the integer case). The combined final income is $141,235$ in the continuous and $141,234$ in the integer case. This represents the optimal tradeoff between strategies (i) and (ii) for the given parameters.

4 OPTIMAL STRATEGY FOR AN UNKNOWN OPPONENT STRATEGY

In the next step, we will consider the non-cooperative game, where communication between the two players is not allowed. The usual approach for such a game is to find a Nash equilibrium. In a Nash equilibrium, the optimal strategy for player A would be the profit maximizing production strategy conditional on the *equilibrium* production strategy of producer B.

By contrast, our goal is to find an optimal strategy for player A, that is, a production schema that gives the highest possible yield from A, no matter what player B is doing. We will prioritize the robustness of the strategy for player A and therefore assume that player B does not necessarily follow a Nash equilibrium production strategy. In particular, we assume that player A has no information on the behavior of player B, and consequently, that every move of player B that respects the per player resource limit, occurs with the same probability.

The valid choice of quantities for one player in each of the six turns is called a trajectory. We define a set S

that consists of n_S trajectories of player B:

$$s_k = \left(q^1_{B,k}, q^2_{B,k}, ..., q^{n_t}_{B,k}\right), \quad k \in \{1, .., n_S\} \tag{4}$$

We assume that all trajectories in S are feasible in the sense that

$$\sum_{t=1}^{n_t} q^t_{B,k} \le s. \tag{5}$$

Now we can define an new objective function as the average over the trajectories in s, assuming that each of the trajectories can occur with the same uniform probability of $1/n_S$:

$$\overline{x}_A = \frac{1}{n_S} \sum_{k \in S} x_A(q_A, q_{B,k}). \tag{6}$$

Then the new optimization problem reads as follows:

$$\begin{aligned} &\underset{q_A}{\text{maximize}} && \overline{x}_A, \\ &\text{subject to} && \sum_{t=1}^{n_t} q^t_A \le s. \end{aligned} \tag{7}$$

The objective here is to maximize the mean income $\overline{x}_A$ of player A. Using the definition of $x_i(q_A, q_B)$, and changing the order of the two sums (the averaging over trajectories and the sum over the rounds), we can write

$$\begin{aligned} \overline{x}_A &= \frac{1}{n_S} \sum_{k \in S} \sum_{t=1}^{n_t} \left(q^t_A \cdot r^{n_t - t} \cdot \left(a - b\left(q^t_A + q^t_{B,k}\right)\right)\right) \\ &= \sum_{t=1}^{n_t} (q^t_A \cdot r^{n_t - t} \cdot (a - b(q^t_A + \underbrace{\frac{1}{n_S} \sum_{k \in S} q^t_{B,k}}_{q^t_{\text{eff}}}))). \end{aligned} \tag{8}$$

We have accumulated the averaging process in an *effective quantity* q_{eff} that expresses the average production. In order to solve (7) in the next step we need to specify the set of trajectories.

We create a set of trajectories S by discretizing the interval of possible quantities at a given time. The maximum quantity that can be produced is equal to the per player resource stock s. We select an integer value n_α. Then the production level for a certain time step can no longer be chosen arbitrarily, but must be an integer multiple of the basic step size $\delta := \frac{s}{n_\alpha - 1}$. That means, for each trajectory $k \in S$ and each time step $t = 1, \dots, n_t$ there exists such a multiplier $\alpha^t_{B,k} \in \{0, 1, 2, ..., n_\alpha - 1\}$, such that $q^t_{B,k} = \delta \alpha^t_{B,k}$. (Note that constraint (5) also needs to be fulfilled, still.)

The total production output of player B over all time periods $t = 1, \dots, n_t$ is also an integer multiplier of δ. For $\alpha \in \{0, \dots, n_\alpha - 1\}$ we denote by $S_\alpha \subseteq S$ those trajectories from S with $\sum_{t=1}^{n_t} q^t_{B,k} = \delta\alpha$.

The cardinality of S_α is denoted by $n_{S_\alpha} := |S_\alpha|$. The value of n_{S_α} can be computed using the binomial coefficient:

$$n_{S_\alpha} = \binom{\alpha + n_t - 1}{n_t - 1} \tag{9}$$

Using the following combinatorial identity:

$$\sum_{k=0}^{m} \binom{n+k}{n} = \binom{n+m+1}{n+1}, \tag{10}$$

we can calculate the number of all possible trajectories

$$\begin{aligned} n_S &= \sum_{\alpha=0}^{n_\alpha - 1} n_{S_\alpha} = \sum_{\alpha=0}^{n_\alpha - 1} \binom{\alpha + n_t - 1}{n_t - 1} \\ &= \binom{n_t + n_\alpha}{n_t} = \frac{(n_t + n_\alpha)!}{n_t! n_\alpha!}. \end{aligned} \tag{11}$$

Consider an arbitrary trajectory $k \in S$ with coefficient vector $(q^1_{B,k}, \dots, q^{n_t}_{B,k})$. Then any permutation of these coefficients leads to feasible trajectory, since the total production does not change by permuting their order. Hence summing the coefficients for any fixed time step t always yields the same constant value:

$$\sum_{k \in S} q^1_{B,k} = \sum_{k \in S} q^2_{B,k} = \ldots = \sum_{k \in S} q^{n_t}_{B,k}, \tag{12}$$

hence

$$\sum_{t=1}^{n_t} \sum_{k \in S} q^t_{B,k} = n_t \cdot \sum_{k \in S} q^1_{B,k} = \ldots = n_t \sum_{k \in S} q^{n_t}_{B,k} \tag{13}$$

follows. In particular $q^1_{\text{eff}} = \ldots = q^{n_t}_{\text{eff}}$, and we simply write q_{eff} in the sequel.

We can express the left-hand side in (13) as follows:

$$\sum_{t=1}^{n_t} \sum_{k \in S} q^t_{B,k} = \sum_{\alpha=0}^{n_\alpha - 1} n_{S_\alpha} \cdot \delta\alpha, \tag{14}$$

and, using the identity of (9), arrive at the following expression for q_{eff}:

$$\begin{aligned} q_{\text{eff}} &= \frac{\delta}{n_t n_S} \sum_{\alpha=0}^{n_\alpha - 1} \alpha \frac{(\alpha + n_t - 1)!}{(n_t - 1)! \alpha!} \\ &= \frac{\delta}{n_S} \sum_{\alpha=0}^{n_\alpha - 1} \frac{(\alpha + n_t - 1)!}{(n_t)!(\alpha - 1)!} \\ &\overset{(10)}{=} \frac{\delta}{n_s} \binom{n_t + n_\alpha}{n_t + 1} \\ &\overset{(11)}{=} \delta \frac{n_t! n_\alpha!}{(n_t + n_\alpha)!} \frac{(n_t + n_\alpha)!}{(n_t + 1)!(n_\alpha - 1)!} \\ &= \frac{\delta n_\alpha}{n_t + 1} = \frac{s}{n_t + 1}. \end{aligned} \tag{15}$$

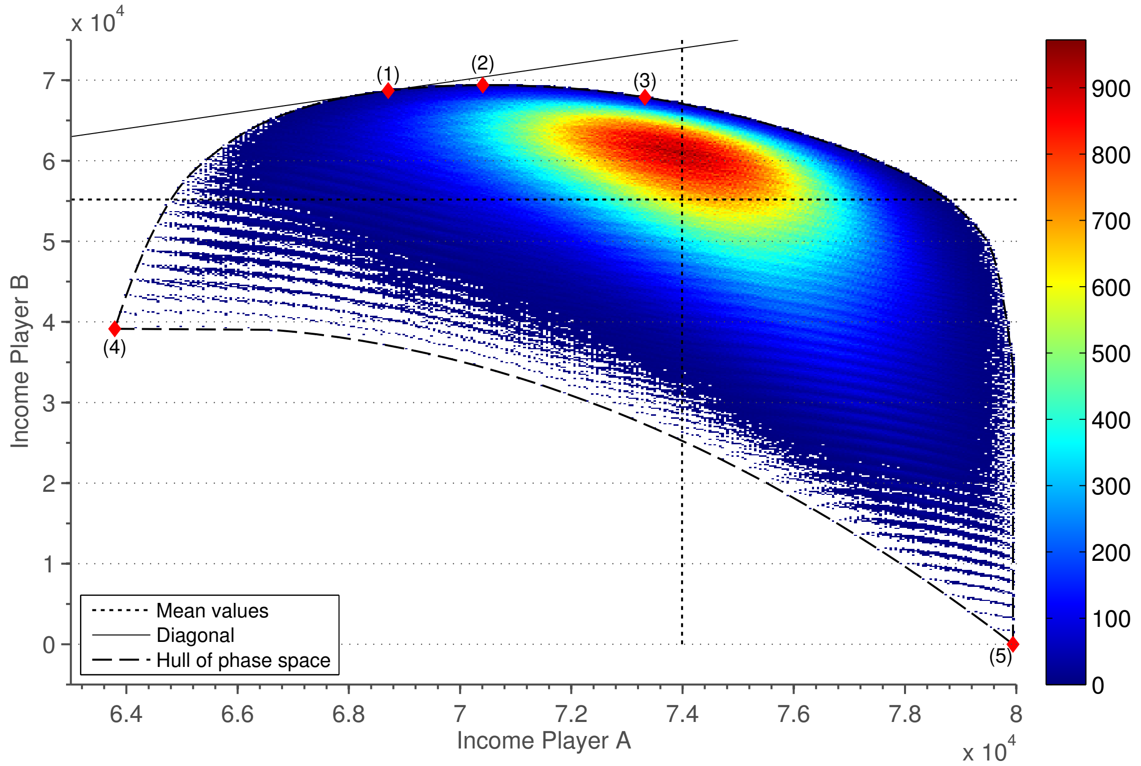

Figure 2: Plot of possible incomes for strategy 4 of player A and an equidstantly discretized set of trajectories for player B with stepsize $170/40$. Marked points are: (1): (68708,68708), symmetric strategy, (2): (70409, 69395) highest income of B, (3): (73325, 67898), highest combined income, (4): (63789, 39151), lowest income of A, (5): (79943, 0), lowest income of B, highest income of A, lowest combined income

We emphasize that we obtained an expression for q_{eff} that does not depend on δ, the step size of the discretization. Accordingly, the averaging over trajectories in the calculation of the total income of player A reduces to:

$$\overline{x}_A = \sum_{t=1}^{n_t} q_A^t \cdot r^{n_t - t} \cdot \left(a - b \left(q_A^t + \frac{s}{n_t + 1} \right) \right). \quad (16)$$

We have shown that the optimal solution of (7) is independent of the stepsize of the discretization. It is also straightforward to show that (16) holds for the case $\delta \to 0$, where in the limit the summation in (7) is replaced by integration.

Using (16), problem (7) is easily accessible with standard nonlinear global optimization techniques. The optimal solution for the continuous and integer cases, obtained by the solver SCIP, are summarized in Table 4.

	q_c^1	q_c^2	q_c^3	q_c^4	q_c^5	q_c^6
C	59.31	47.87	35.28	21.36	6.17	0.00
I	59.00	48.00	35.00	22.00	6.00	0.00

Table 4: Optimal strategy (production schema) for player A in the case of an unknown strategy for player B, in the integer (I) and continuous (C) case.

The mean incomes achieved by both players, when player A always uses the integer solution, are $(\overline{x}_A, \overline{x}_B) = (7.3990 \cdot 10^4, 5.5192 \cdot 10^4)$.

5 DISCUSSION

In the previous section, we derived a strategy for player A, that will lead to his highest average income, if the opponent makes random decisions. In this section, the properties of that solution will be discussed.

We will assume, that we have advised player A to strictly follow the strategy shown in Table 4. The averaging process implies that the game is repeated an infinite number of times, however this will not be the case in practice, therefore we will consider the possible scenarios that can occur in a single game under the given assumptions.

To allow for a visual interpretation, a diagram of the phase space of the game is shown in Figure 2. The diagram shows the incomes of players A and B, where player A uses the integer strategy in Table 4, and player B uses one of $10.7 \cdot 10^6$ trajectories that result from an again equidistant discretization of all possible trajectories with step-size $\delta = 170/40$. The number of points (i.e., trajectories) at a given pixel are represented by shades of color.

Some extreme points are marked in Figure 2. The lower right corner of the space is defined by the point where player B achieves zero income with the trajectory $q_B^1 = \ldots = q_B^6 = 0$. At the same time, this is the point where x_A is at its maximum, since player A achieves the highest possible prices. This is also the point, where the combined income of both players has the lowest value. The left corner of the phase space is marked by the minimum of x_A. The corresponding trajectory of player B is $q_B = (170, 0, 0, 0, 0, 0)$, which means that player B sells all his products in the first round. Due to the interest element of the model, this is the biggest possible disturbance of player A. However, this strategy is not beneficial for player B, as it yields a sub average income.

The highest combined income of $141,224$ is achieved when player B follows the trajectory $q_B = (25.5, 25.5, 29.75, 29.75, 34.00, 25.5)$. The achieved combined incomes are very close to the optimal solution in the cooperative case, however the income of player A is 8% higher than that of player B.

Player B achieves his highest possible income with the trajectory $q_B = (46.75, 38.25, 34.00, 25.50, 21.25, 4.25)$. This results in a sub average income for player A, however, player A still has a higher income than player B. In fact, we find that the every point of the phase space is below the diagonal. In other words, if player A follows the strategy in Table 4, there is no way for player B to achieve a higher income. Due to the symmetry of the model, it is of course possible for player B to achieve exactly the same income, by using the same strategy as player A.

From Figure 2, it is also clear that both players following the strategy in Table 4 cannot be a Nash equilibrium, since a different trajectory is optimal for player B. Van Veldhuizen and Sonnemans [8] derive the (feedback) Nash equilibrium for the same parameters using a backward induction type procedure. Our results show that the strategy in Table 4 means that player A will overproduce relative to the Nash equilibrium. Intuitively, player B will on average produce less than the Nash equilibrium in earlier periods and more in later periods, meaning it is optimal for player A to shift his production to earlier periods in response.

In a given instance of the game, there is still some room for improvement of our advice for player A, by using the information gained each round on the choices of player B. This leads to an iterative approach, where we first calculate the strategy in Table 4, and advise player A to choose the corresponding quantity in the first round. When player B has made his move, we can derive a new upper bound s, for player B, based on the amount of his product he has sold in the first round. Based on this, we can compute a *refined* strategy for the second round. We can iterate these steps each round, deriving a strategy that dynamically uses all the available information.

6 CONCLUSIONS

In this paper, we took a statistical approach to finding an optimal strategy for a two-player game that arises in the context of nonrenewable resources and the Hotelling rule. By stochastical considerations we reformulated a potentially large optimization problem to one that can be quickly solved to optimality using a standard branch-and-bound approach. Using the reformulated problem, we computed a strategy, that is unbeatable by the opponent in the sense that the opponent is not able to achieve a higher income, no matter what production strategy he might follow.

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